Rasyonel Form:

A $n \times n$ verildiğinde, karakteristik matris $\mathbb{R}$ üzerinde 1'den büyük dereceli asal polinomların çarpımı olabilir.

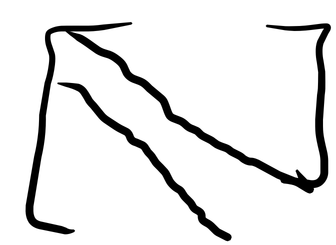
x^2+1 polinomunun $\mathbb{R}$ içinde kökü yoktur. Bu tip polinomları içeren karakteristik polinomlar bulduğumuzda A için A'ya benzer kanonik biçim Rasyonel Formdur.

$A \sim B \sim$ Kanonik Form. → Rasyonel
→ Jordan
→ Smith N. F.

R.F.:

a) $f = a_0 + a_1 x + a_2 x^2 + \dots + a_{m-1} x^{m-1} + \underline{\underline{x^m}}$, R üzerinde monik bir polinom olsun. Aşağıda verilen C_f matrisine f in eşlikçi matrisi denir

$$C_f = \begin{bmatrix} 0 & & & & & -a_0 \\ 1 & 0 & & & & -a_1 \\ 0 & 1 & 0 & \cdots & & -a_2 \\ 0 & 0 & 1 & 0 & & \vdots \\ & & & \ddots & & \vdots \\ 0 & \cdots & & 0 & 1 & -a_{m-1} \end{bmatrix}_{m \times m}$$



Bu durumda

$$x \cdot I - C_f = \begin{bmatrix} x & & & & a_0 \\ -1 & x & & & a_1 \\ 0 & -1 & x & & a_2 \\ & & \ddots & \ddots & \vdots \\ & & & & a_{m-1} \\ 0 & \cdots & 0 & -1 & x + a_{m-1} \end{bmatrix}$$

ve $\Delta_m = (-1)^{1+m} \cdot a_0 \cdot (-1)^{m-1} + (-1)^{2+m} \cdot a_1 (-1)^{m-2} \cdot x$

m. sütun kofaktör

$+ (-1)^{3+m} \cdot a_2 \cdot (x^2) (-1)^{m-3} + \dots + (-1)^{m+m} \cdot (x + a_{m-1}) \cdot x^{m-1}$

$= a_0 + a_1 x + a_2 x^2 + \dots + a_{m-1} x^{m-1} + x^m = f$

$$\Delta_n = f = |xI - C_f|$$

$$\Delta_{n-1} = \text{ebob}\left\{ \underbrace{\begin{vmatrix} -1 & x & & \\ 0 & -1 & x & \\ 0 & 0 & -1 & \\ & & & \ddots \\ & 0 & 0 & -1 \end{vmatrix}}_{} = (-1)^{n-1}, \ldots, \right\}$$

$$= 1$$

$$d_n = \frac{\Delta_n}{\Delta_{n-1}} = f \quad , \quad d_{n-1} = \cdots = d_1 = 1$$

$a_1)$ $(xI - C_f)$ in normal formu

$$D_f = \begin{bmatrix} 1 & & & \\ & 1 & & 0 \\ & & 1 & \\ 0 & & & \ddots \\ & & & & f \end{bmatrix}$$ dir.

$a_2)$ $(xI - C_f)$ in değişmez çarpanları

$d_1 = \dots = d_{n-1} = 1$, $d_n = f$ tir.

$a_3)$ $f(x) = \underline{\underline{p_1}}^{e_1} \cdot \underline{\underline{p_2}}^{e_2} \cdots \underline{\underline{p_r}}^{e_r}$;

p_i'ler farklı asal polinomlar ve e_i ler pozitif olsun. $(xI - C_f)$ in temel bölenleri $p_1^{e_1}, p_2^{e_2}, \dots, p_r^{e_r}$ dir.

$D_{p_1^{e_1}} = \text{köşeg}\,[1, 1, \ldots, 1, p_1^{e_1}]$ alırsak

$a_4)$ $xI - C_f$

$$D' = \begin{bmatrix} D_{p_1^{e_1}} & & 0 \\ & D_{p_2^{e_2}} & \\ 0 & & \ddots \quad D_{p_r^{e_r}} \end{bmatrix} \qquad D_f$$

matrisine denk olur.

$$\Delta'_m = |D_{p_1^{e_1}}| \cdot |D_{p_2^{e_2}}| \cdots |D_{p_r^{e_r}}| = p_1^{e_1} \cdots p_r^{e_r}$$

$$\Delta'_m = f$$

$$\Delta'_{m-1} = \left\{ \begin{bmatrix} \begin{matrix} 1 & & \\ & \ddots & \\ & & p_1^{e_1} \end{matrix} & \\ & \begin{bmatrix} D_{p_2}^{e_2} & & \\ & \ddots & \\ & & D_{p_r}^{e_r} \end{bmatrix} \end{bmatrix} \right.,$$

$$p_2^{e_2} \cdot p_3^{e_3} \cdots p_r^{e_r} = \frac{f}{p_1^{e_1}}, \quad \frac{f}{p_2^{e_2}}$$

$$\cancel{p_1^{e_1}} \cdot p_2^{e_2} \cdots p_r^{e_r}, \quad p_1^{e_1} \cancel{p_2^{e_2}} \cdots p_r^{e_r}$$

$$\frac{f}{p_3^{e_3}} \cdots$$

$$\Delta'_{m-1} = 1$$

D' matrisinin değişmez çarpanları

$$d'_m = \frac{\Delta'_m}{\Delta'_{m-1}} = f \quad , \quad d'_1 = \cdots = d'_{m-1} = \frac{1}{\quad} \quad \frac{\Delta'_{m-1}}{\Delta'_{m-2}} \quad \begin{matrix} 1 \\ 1 \end{matrix}$$

D' değişmez çarpanları ile $xI - C_f$ in değişmez çarpanları eşit; ve bu iki matris birbirine denktir. (a_4 ispatlandı.)

b) $f_1, f_2, \dots, f_s$; $\quad \operatorname{der} f_i = m_i \geq 1$

ve $f_{i-1} \mid f_i$:

$$R = \begin{bmatrix} C_{f_1} & & & \\ & C_{f_2} & & \\ & & \ddots & \\ & & & C_{f_s} \end{bmatrix}$$

$1 \cdots 1_{f_1}$

$1 \cdots 1_{f_2}$

$$\operatorname{der} f_1 + \operatorname{der} f_2 + \dots + \operatorname{der} f_s = n$$

$$m_1 + m_2 + \dots + m_s = \underline{\underline{n}}$$

Yukarıdaki R matrisine Rasyonel Formdadır (Biçimdedir) denir.

(a_1) den her $(xI_{m_i} - C_{f_i})$ nin

$\text{köşeg}[1, \ldots, 1, f_i]$ matrisine denk i

bu yüzden $(xI - C)$ nin

$\text{köşeg}[1, 1, \ldots, 1, f_1, f_2, \ldots, f_s]$

matrisine denkliğini biliyoruz.

Böylece R rasyonel formda ise,

$(xI - R)$ nin değişmez çarpanları

$d_1 = \ldots = d_s = 1$; $d_{s+1} = f_1$, $d_{s+2} = f_2, \ldots, d_{s+(n-s)}$

$\qquad \parallel$

$d_n = f_s$

olur $\quad C \sim \begin{bmatrix} d_1 & & \\ & \ddots & \\ & & d_s \end{bmatrix} \sim$

$xI - C \quad xI - R \qquad R = \begin{bmatrix} C_{d_1} & & \\ & \ddots & \\ & & C_{d_s} \end{bmatrix}$

$x - a \quad \ldots \quad x - a_i$

$f_i = \underline{(p_{i1}^{k_{i1}}) \, p_{i2}^{k_{i2}} \cdots p_{it_i}^{k_{it_i}}}$, f_i nin

asal polinomlara ayrılışı olsun.

$$\boxed{\begin{array}{c} b^2 - 4ac < 0 \\ ax^2 + bx + c \end{array}}$$

$(x-a)^{b_i}$

Bir yandan (a_4) ile her bir

$xI_{m_i} - C_{f_i}$,

$$D_i = \begin{bmatrix} D_{p_{i1}}^{k_{i1}} & & \\ & \ddots & \\ & & D_{p_{it_i}}^{k_{it_i}} \end{bmatrix}$$

matrisine denk; ve $xI - C$,

$$D' = \begin{bmatrix} D_1 & & & \\ & D_2 & & \\ & & \ddots & \\ & & & D_s \end{bmatrix}$$

matrisine denktir.

Diğer yandan $(xI - C)$ nin

temel bölenleri $\Big\{ \underbrace{p_{i1}^{k_{i1}}, \ldots, p_{ij}^{k_{ij}}}_{f_i} \;\Big|\; \begin{array}{l} i = 1, \ldots, s \\ j = 1, \ldots, n_i \end{array} \Big\}$

terimlerinin yer değiştirmesiyle elde edilir.

Böylece $(xI - C)$,

$S = \{ p_i^{e_{ij}} \; 1 \le i \le s, \; 1 \le j \le n_i \}$ temel bölenleri

olmak üzere

$$D = \begin{bmatrix} D_{p_1}^{e_{11}} & & & & & \\ & \ddots & & & & \\ & & D_{p_1}^{e_{1n_1}} & & & \\ & & & \ddots & & \\ & & & & D_{p_r}^{e_{11}} & & \\ & & & & & \ddots & \\ & & & & & & D_{p_r}^{e_{1n_r}} \end{bmatrix}$$

matrisine denktir.

$D_{p_i^{e_{ij}}}$, $(xI - C_{p_i^{e_{ij}}})$ nin normal formu olduğundan (a_1) ile $xI - C$ matrisi

$$R_A = \begin{bmatrix} C_{p_1^{e_{11}}} & & & & & \\ & \ddots & & & & \\ & & C_{p_1^{e_{1n_1}}} & & & \\ & & & \ddots & & \\ & & & & C_{p_r^{e_{r1}}} & \\ & & & & & \ddots & \\ & & & & & & C_{p_r^{e_{rn_r}}} \end{bmatrix}$$

matrisine denktir.

b_2) C matrisi ile R_A benzerdir.

Yukarıdaki yazılıştaki R_A matrisine

Asal Rasyonel Form denir

$$R = \begin{bmatrix} C_{f_1} & & & \\ & C_{f_2} & & \\ & & \ddots & \\ & & & C_{f_s} \end{bmatrix}$$

$d_1 = f_1 = p_{11}^{e_{11}} p_{12}^{e_{12}}$

$d_2 = f_2 = p_{21}^{e_{21}} \cdot p_{22}^{e_{22}}$

$\vdots$

$d_s = f_s =$

$$R_A = \begin{bmatrix} \begin{bmatrix} C_{p_1^{e_{11}}} & & \\ & C_{p_1^{e_{12}}} & \\ & & \ddots \end{bmatrix} & \\ & \end{bmatrix}$$

$e_{i1} \geq e_{i2} \geq \dots \geq e_{ir_i}$

şartıyla tek türlü belirlidir

Örnek: A, $\mathbb{R}$ üzerinde kare bir matris ve

$\underline{\Delta_A = |xI - A|}$ karakteristik polinomu

$\Delta_A = (x+1)^2 \cdot (x-2)^3$ olsun. $(xI - A)$ nın

temel bölenleri aşağıdakilerden biridir:

a) $\underbrace{(x+1)^2}, \underbrace{(x-2)^3}$

$$\begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & (x+1)^2(x-2)^3 \end{bmatrix}$$

b) $(x+1), \underbrace{(x+1)}, \underbrace{(x-2)^3}$

$$\begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & x+1 & \\ & & & & \underline{(x-2)^3(x+1)} \end{bmatrix}$$

c) $(x+1)^2, (x-2)^2, (x-2)$

$$\begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & x-2 & \\ & & & & (x-2)^2(x+1)^2 \end{bmatrix}$$

temel

a) $\underbrace{(x+1)^2}, \underbrace{(x-2)^3}$

$$\begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & (x+1)^2(x-2)^3 \end{bmatrix}$$

$$\begin{bmatrix} C_{(x+1)^2} & 0 \\ 0 & C_{(x-2)^3} \end{bmatrix} = \begin{bmatrix} \begin{matrix} 0 & -1 \\ 1 & -2 \end{matrix} & \\ & \begin{matrix} 0 & 0 & 8 \\ 1 & 0 & -6 \\ 0 & 1 & 12 \end{matrix} \end{bmatrix}$$

$x^3 - 3 \cdot 4 \cdot x^2 + 3 \cdot 2 \cdot x - 8$

d) $(x+1), (x+1), (x-2), (x-2), (x-2)$

$$\begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & (x-2) & & \\ & & & (x+1)(x-2) & \\ & & & & (x+1)(x-2) \end{bmatrix}$$

e) $(x+1)^2, (x-2), (x-2), (x-2)$

$$\begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & (x-2) & & \\ & & & (x-2) & \\ & & & & (x-2)(x+1)^2 \end{bmatrix}$$

$$R(C) = \begin{bmatrix} C_{x-2} & & 0 \\ & C_{x-2} & \\ 0 & & C_{(x-2)(x+1)^2} \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & (x-2) & & \\ & & & (x-2) & \\ & & & & (x-2)(x+1)^2 \end{bmatrix}$$

$$= \left[\begin{array}{c|c|ccc} 2 & & & & \\ \hline & 2 & & & \\ \hline & & 0 & 0 & 2 \\ & & 1 & 0 & 3 \\ & & 0 & 1 & 0 \end{array}\right]$$

$$(x-2)(x^2+2x+1)$$

$$= x^3 - 3x - 2$$

$$\underset{p_1^{a_1}}{(x-2)^1},\ \underset{p_1^{a_1}}{(x-2)^1},\ \underset{p_1^{a_1}}{(x-2)^1}\underbrace{(x+1)^2}_{p_2^{a_2}}$$

$$\left[\begin{array}{ccc|cc} 2 & 0 & 0 & & \\ 0 & 2 & 0 & & \\ 0 & 0 & 2 & & \\ \hline & & & 0 & -1 \\ & & & 1 & -2 \end{array}\right]$$