Örnek: $A = \begin{bmatrix} x^2 & x+1 & 0 \\ x^2-1 & x+1 & 0 \\ 0 & 0 & 3(x+1)^2 \end{bmatrix}$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & x+1 & 0 \\ 0 & 0 & (x+1)^2 \end{bmatrix} = \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix}$$

$$d_1 = \frac{\Delta_1}{\Delta_0}, \quad d_2 = \frac{\Delta_2}{\Delta_1}, \quad d_3 = \frac{\Delta_3}{\Delta_2}$$

Δ_i = A nın $i \times i$ minörlerinin e.b.o.b u.

$$\Delta_1 = \text{ebob}\left\{ \underline{\underline{x^2}},\ x+1,\ \underline{\underline{x^2-1}},\ 3(x+1)^2 \right\} = 1$$

$$\Delta_0 = 1$$

$$\Delta_2 = \text{ebob}\left\{ \underbrace{\begin{vmatrix} x^2 & x+1 \\ x^2-1 & x+1 \end{vmatrix}}_{(x+1)},\ \underbrace{\begin{vmatrix} x^2 & 0 \\ x^2-1 & 0 \end{vmatrix}}_{0},\ \underbrace{\begin{vmatrix} x+1 & 0 \\ x+1 & 0 \end{vmatrix}}_{0} \right.$$

$$A = \begin{bmatrix} x^2 & x+1 & 0 \\ x^2-1 & x+1 & 0 \\ 0 & 0 & 3(x+1)^2 \end{bmatrix}$$

Δ_2 devam... $\begin{vmatrix} x^2 & x+1 \\ 0 & 0 \end{vmatrix},\ \overbrace{\begin{vmatrix} x^2 & 0 \\ 0 & 3(x+1)^2 \end{vmatrix}}^{x+1},\ \overbrace{\begin{vmatrix} x+1 & 0 \\ 0 & 3(x+1)^2 \end{vmatrix}}^{x+1}$

$\underbrace{\begin{vmatrix} x+1 & 0 \\ 0 & 3(x+1)^2 \end{vmatrix}}_{(x+1)},\ \begin{vmatrix} x^2-1 & x+1 \\ 0 & 0 \end{vmatrix},\ \underbrace{\begin{vmatrix} x^2-1 & 0 \\ 0 & 3(x+1)^2 \end{vmatrix}}_{(x+1)}$

$\Delta_0 = 1$, $\Delta_1 = 1$, $\Delta_2 = x+1$, $\Delta_3 = 3(x+1)^2$

$$\cdot\left(\cancel{x^3}+\cancel{x^2}-\left(\cancel{x^3}+\cancel{x^2} - x - 1\right)\right)$$

$$\Delta_3 = 3\cdot(x+1)^2(x+1)\cdot 1$$

$$\boxed{\cancel{\begin{array}{l} 4 \times 4 \\ 1 + \binom{4}{3}\binom{4}{3} \\ + \binom{4}{2}\binom{4}{2} + \binom{4}{1}^2 \end{array}}}$$

$$d_3 = \frac{\Delta_3}{\Delta_2} = \frac{(x+1)^3}{(x+1)} = (x+1)^2$$

$$A \sim D_A = \begin{bmatrix} 1 & & 0 \\ & x+1 & \\ 0 & & (x+1)^2 \end{bmatrix}$$

Matris Katsayılı Polinomlar:

$$M \in R[x]^{3\times 3} \quad \text{ve} \quad M = \begin{bmatrix} x+1 & -x^2+2x & 0 \\ -5+x^3 & x^3+x^2 & x-1 \\ 0 & 1-x^3 & x^2+2x \end{bmatrix}$$

$$M = 1 \cdot \begin{bmatrix} 1 & 0 & 0 \\ -5 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} + x \cdot \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

$$+ x^2 \cdot \begin{bmatrix} 0 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + x^3 \cdot \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & -1 & 0 \end{bmatrix}$$

Lemma 0: $\mathbb{R}^{n\times n}$ de bir A matrisi ve her i için $B_i \in \mathbb{R}^{n\times n}$ olmak üzere

$$B = B_0 + x \cdot B_1 + x^2 \cdot B_2 + \dots + x^d B_d$$

matrisi verildiğinde

$$B = (xI - A) \cdot Q + R, \qquad R \in \mathbb{R}^{n\times n}$$

olacak şekilde tek türlü belirli $Q \in \mathbb{R}[x]^{n\times n}$ ve R vardır.

Özel olarak

$$B = (xI - A) \cdot Q$$

olması için gerek ve yeter şart

$$B_0 + A.B_1 + A^2.B_2 + \dots + A^d.B_d = 0$$

olmasıdır.

Lemma: A ve B, R üzerinde $n\times n$ matrisler olsun.

$$(xI - B) = P.(xI - A).Q \; ; \quad P, Q \in R[x]^{n\times n}$$

olsun. Eğer

$$P = P_0 + xP_1 + x^2.P_2 + \dots + x^r.P_r, \quad P_i \in R^{n\times n}$$

ise;

(i) $\tilde{P} = P_0 + B \cdot P_1 + B^2 \cdot P_2 + \dots + B^r \cdot P_r$

tersinirdir.

$$\boxed{P^{-1} A P = B}$$

(ii) $\underline{\tilde{P} \cdot A \cdot \tilde{P}^{-1} = B}$ dir.

İspat: (i) – a) $\tilde{P} \cdot A = B \cdot \tilde{P}$ dir.

P ve $Q \in \mathbb{R}[x]^{n \times n}$ tersinir matrislerdir:

$$xI - B = P \cdot (xI - A) \cdot Q$$

$$\underbrace{|xI - B|}_{\text{n. derece monik polinom}} = |P| \cdot \underbrace{|xI - A|}_{\text{n. der. monik}} \cdot |Q|$$

$\Rightarrow |P|.|Q| = 1 \Rightarrow |P| \neq 0$ ve $|Q| \neq 0$

Böylece P ve Q tersinirdir.

$$(xI - B) = P.(xI - A).Q$$

$$P.(xI - A) = (xI - B).Q^{-1}$$

$$x.P - P.A = x.P - \sum_{i=0}^{r} x^i.P_i.A = (xI - B)Q^{-1}$$

Lemma①

$$\Rightarrow \underbrace{B\Big(\sum_{i=0}^{r} B^i.P_i\Big)}_{\tilde{P}} - \underbrace{\sum_{i=0}^{r} B^i.P_i.A}_{\tilde{P}} = 0$$

$$B.\tilde{P} = \tilde{P}.A$$

$$B.\tilde{P}A = \tilde{P}A^2$$

$$B.B\tilde{P} = \tilde{P}.A^2$$

$$B^2.\tilde{P} = \tilde{P}.A^2$$

her $j \geq 1$ için $B^j.\tilde{P} = \tilde{P}.A^j$

$$B.\left(\sum_{i=0}^{r} B^i.P_i\right) - \sum_{i=0}^{r} B^i.P_i\,A = 0$$

P ve Q nun tersinir olduğu görüldü;

$P^{-1} = P_0' + x.P_1' + x^2.P_2' + \ldots + x^k.P_k'$ olsun,

$$I = P \cdot P^{-1} = \left(\sum_{i=0}^{r} x^i P_i\right)\left(\sum_{j=0}^{k} x^j \cdot P_j'\right)$$

$$= \sum_{i=0}^{r} \sum_{j=0}^{k} x^{i+j} \cdot P_i P_j' = I$$

$\tilde{P} A^j = B^j \cdot \tilde{P}$ uygulanarak

$$\tilde{P} \cdot \underbrace{\left(\sum_{j=0}^{k} A^j P_j'\right)} = \sum_{j=0}^{k} \underbrace{\tilde{P} \cdot A^j} P_j' =$$

$$= \sum_{j=0}^{k} B^j \cdot \tilde{P} \cdot P_j' = \sum_{j=0}^{k} B^j \left(\sum_{i=0} B^i \cdot P_i\right) \cdot P_j'$$

$$= \sum_{j=0}^{k} \sum_{i=0}^{r} B^{j+i} \cdot P_i \cdot P_j' = I$$

$\Rightarrow \tilde{P}$ tersinir çıkar.

(ii) için $\tilde{P} A = B . \tilde{P}$ bulunmuştu.

$\tilde{P} . A . \tilde{P}^{-1} = B$ olur.

Örnek: 1) $\mathbb{Q}[x]$ te aşağıda verilen her bir matrisi normal formuna indirgeyiniz. Değişmez çarpanlarını ve temel bölenlerini bulunuz.

a) köşeg $[x^2-1,\ 1,\ x-1,\ 2]$

$$= \begin{bmatrix} x^2-1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & x-1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \overset{S_1 \leftrightarrow S_2}{\sim} \begin{bmatrix} 0 & 1 & 0 & 0 \\ x^2-1 & 0 & 0 & 0 \\ 0 & 0 & x-1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

$$C_1 \leftrightarrow C_2 \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & x^2-1 & 0 & 0 \\ 0 & 0 & x-1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \overset{S_2 \leftrightarrow S}{\sim} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & x-1 & 0 \\ 0 & x^2-1 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ x^2-1 & 0 & 0 & 0 \\ 0 & 0 & x-1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

$$C_2 \leftrightarrow C_4 \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & x-1 & 0 \\ 0 & 0 & 0 & x^2-1 \end{bmatrix} \overset{\frac{1}{2} \cdot C_2}{\sim} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & x-1 & 0 \\ 0 & 0 & 0 & x^2-1 \end{bmatrix}$$

$d_1 = d_2 = 1, \; d_3 = x-1, \; d_4 = x^2-1$ $\qquad d_1 \mid d_2, \; d_2 \mid d_3, \; d_3 \mid d_4$

$d_1 = d_2 = 1,\ d_3 = x-1,\ d_4 = x^2-1$

A'nın değişmez çarpanları

$\{x-1,\ x-1,\ x+1\}$ temel bölenleri.

Örnek: $A = \text{köşeg}[x-1,\ x+1,\ x-1]$

$$A = \begin{bmatrix} x-1 & 0 & 0 \\ 0 & x+1 & 0 \\ 0 & 0 & x-1 \end{bmatrix} \underset{\sim}{S_2+S_1 \to S_1} \begin{bmatrix} x-1 & x+1 & 0 \\ 0 & x+1 & 0 \\ 0 & 0 & x-1 \end{bmatrix}$$

$$\underset{\sim}{-C_2+C_1} \begin{bmatrix} -2 & x+1 & 0 \\ -x-1 & x+1 & 0 \\ 0 & 0 & x-1 \end{bmatrix} \underset{\sim}{\frac{-1}{2}S_1} \begin{bmatrix} 1 & \frac{-x-1}{2} & 0 \\ -x-1 & x+1 & 0 \\ 0 & 0 & x-1 \end{bmatrix}$$

$$\underset{\sim}{(x+1)S_1+S_2} \begin{bmatrix} 1 & -\frac{x-1}{2} & 0 \\ 0 & (x+1)-\frac{(x+1)^2}{2} & 0 \\ 0 & 0 & x-1 \end{bmatrix} \underset{\sim}{2S_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \boxed{2\cdot(x+1)-(x+1)^2} & 0 \\ 0 & 0 & x-1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & (x+1)(2-x-1) & 0 \\ 0 & 0 & x-1 \end{bmatrix} \overset{-S_2}{\sim} \begin{bmatrix} 1 & 0 & 0 \\ 0 & (x-1)(x+1) & 0 \\ 0 & 0 & x-1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \boxed{\begin{matrix} 2\cdot(x+1) \\ -(x+1)^2 \end{matrix}} & 0 \\ 0 & 0 & x-1 \end{bmatrix}$$

$$\begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix}$$

$$\wr$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & x-1 & 0 \\ 0 & 0 & x^2-1 \end{bmatrix}$$

d_i ler monik $\qquad d_i \mid d_{i+1}$

Değişmez çarpanlar: $1, (x-1), (x^2-1)$

Temel Bölenler: $x-1, x-1, x+1$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & x-1 & 0 \\ 0 & 0 & x^2-1 \end{bmatrix}$$

Örnek: $A = \begin{bmatrix} x & 0 & -2 \\ -1 & x & -1 \\ 0 & -1 & x-2 \end{bmatrix}$

$$\xrightarrow{-S_2 \to S_2} \begin{bmatrix} x & 0 & -2 \\ 1 & -x & 1 \\ 0 & -1 & x-2 \end{bmatrix} \overset{S_1 \leftrightarrow S_2}{\sim} \begin{bmatrix} 1 & -x & 1 \\ x & 0 & -2 \\ 0 & -1 & x-2 \end{bmatrix}$$

$$\begin{matrix} x.C_1 + C_2 \\ \sim \\ -C_1 + C_3 \end{matrix} \begin{bmatrix} 1 & 0 & 0 \\ \boxed{x} & x^2 & -2-x \\ \boxed{0} & -1 & x-2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & x^2 & -2-x \\ 0 & -1 & x-2 \end{bmatrix}$$

$$-S_1 \to S_3 \begin{bmatrix} 1 & 0 & 0 \\ 0 & x^2 & -2-x \\ 0 & 1 & 2-x \end{bmatrix} \quad S_2 \leftrightarrow S_3 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2-x \\ 0 & x^2 & -2-x \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & x^2 & -2-x \\ 0 & -1 & x-2 \end{bmatrix}$$

$$(x-2)C_2 + C_3 \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & x^2 & -2-x+x^2(x-2) \end{bmatrix}$$

$$\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & x^3-2x^2-x-2 \end{matrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & x^2 & -2-x+x^2(x-2) \end{bmatrix}$$

$$\sim \begin{bmatrix} 1\ d_1 & 0 & 0 \\ 0 & 1\ d_2 & 0\quad d_3 \\ 0 & 0 & x^3-2x^2-x-2 \end{bmatrix}$$

değişmez çarpanlar

Örnek : $A = \begin{bmatrix} 1 & 0 & x & -1 \\ x & 1 & x^2 & -x \\ 0 & 0 & x-1 & 0 \\ x^2 & 0 & x^3 & -1 \end{bmatrix}, \quad D_A = ?$

$$A \underset{-x^2 \cdot S_1 + S_4}{\overset{-x \cdot S_1 + S_2}{\sim}} \begin{bmatrix} 1 & 0 & x & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & x-1 & 0 \\ 0 & 0 & 0 & x^2-1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & x-1 & 0 \\ 0 & 0 & 0 & x^2-1 \end{bmatrix} = DA$$

$$\begin{bmatrix} 1 & 0 & \boxed{x \quad -1} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & x-1 & 0 \\ 0 & 0 & 0 & x^2-1 \end{bmatrix}$$

$-x$

1

Örnek: *) aşağıdakilerden hangileri $\mathbb{R}[x]$ üzerinde bir matrisin değişmez çarpanlarıdır?

a) $1, x, x, x^2, x^2+x, x^3-x^2$

$$\begin{bmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_r \end{bmatrix}$$

d_i : monik ve

$d_i \mid d_{i+1}$

$$\begin{bmatrix} 1 & & & \\ & x & & \\ & & x & \\ & & & x^2 \end{bmatrix}$$

$x^2 \nmid x^2+x$ olduğundan (a) da verilenler bir mat. değişmez çarpanları değildir.

b) $1, 1, x, x, x^2, x^3 - x^2, x^4 - x^3$

1

1

x

x

x^2

$x^3 - x^2$

$x^4 - x^3$ ✓

c) $1, 1, -x, x^2, x^2 - x^3$

Terimler birbirini bölmesine rağmen monik olmayan terimler bulunduğundan bir matrisin değişmez çarpanları değildir

d) x^4+1 ✓

Örnek: Aşağıdakilerden hangileri sıfırdan farklı 4 değişmez çarpana sahip bir matrisin temel bölenleridir?

a) x^2, x, $(x-1)^2$, $(x-1)^2$, $x+1$

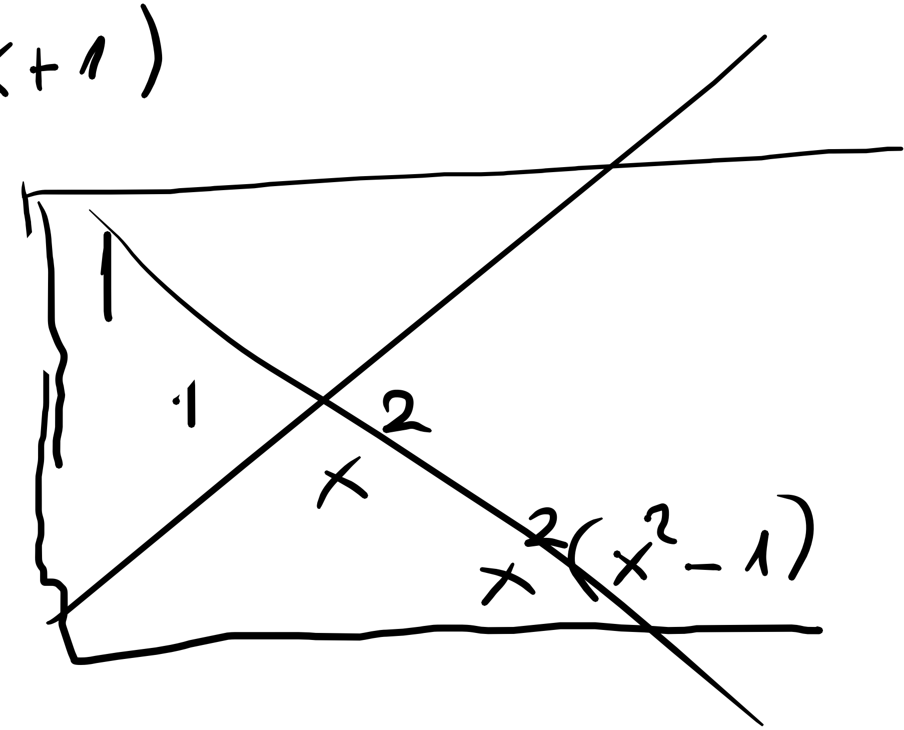
1

1

x

$x^2(x-1)^2(x-1)^2(x+1)$

b) x^2, x^2, $\underbrace{(x^2-1)}_{(x-1)(x+1)}$



$x-5$ $\quad\quad x^2-25$

$(x-5), \quad (x+5)$

$\underline{(x^2-25)}^4 = (x-5)^4 \cdot (x+5)^4$

$(x-5)^4, \quad (x+5)^4$