Rasyonel Form:

A $n \times n$ verildiğinde, karakteristik matris $\mathbb{R}$ üzerinde 1'den büyük dereceli asal polinomların çarpımı olabilir.

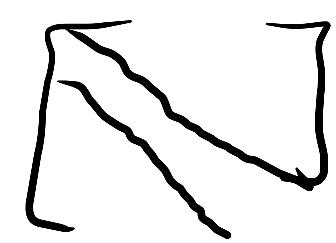
x^2+1 polinomunun $\mathbb{R}$ içinde kökü yoktur. Bu tip polinomlar içeren karakteristik polinomlar bulduğumuzda A için A'ya benzer kanonik biçim Rasyonel Formdur.

A ∼ B ∼ Kanonik Form. → Rasyonel, Jordan, Smith N. F.

R.F.:

a) $f = a_0 + a_1 x + a_2 x^2 + \dots + a_{m-1} x^{m-1} + \underline{\underline{x^m}}$, R üzerinde monik bir polinom olsun. Aşağıda verilen C_f matrisine f in eşlikçi matrisi denir

$$C_f = \begin{bmatrix} 0 & & & & & \vdots & -a_0 \\ 1 & 0 & & & & \vdots & -a_1 \\ 0 & 1 & 0 & \dots & & \vdots & -a_2 \\ 0 & 0 & 1 & 0 & & \vdots & \vdots \\ & & & \ddots & & \vdots & \vdots \\ 0 & \dots & & 0 & 1 & \vdots & -a_{m-1} \end{bmatrix}_{m \times m}$$



Bu durumda

$$x \cdot I - C_f = \begin{bmatrix} x & & & & & a_0 \\ -1 & x & & & & a_1 \\ 0 & -1 & x & & & a_2 \\ & & & \ddots & & \vdots \\ & & & & \ddots & a_{m-1} \\ 0 & \cdots & & 0 & -1 & x + a_{m-1} \end{bmatrix}$$

ve $\Delta_m = (-1)^{1+m} \cdot a_0 \cdot (-1)^{m-1} + (-1)^{2+m} \cdot a_1 (-1)^{m-2} \cdot x$

m. sütun kofaktör

$$+ (-1)^{3+m} \cdot a_2 \cdot (x^2)(-1)^{m-3} + \cdots + (-1)^{m+m} \cdot (x + a_{m-1}) \cdot x^{m-1}$$

$$= a_0 + a_1 x + a_2 x^2 + \cdots + a_{m-1} x^{m-1} + x^m = f$$

$$\Delta_n = f = |xI - C_f|$$

$$\Delta_{n-1} = \text{ebob}\left\{ \underbrace{\begin{vmatrix} -1 & x & & & \\ 0 & -1 & x & & \\ 0 & 0 & -1 & & \\ & & & \ddots & \\ & 0 & & 0 & -1 \end{vmatrix}}_{} = (-1)^{n-1}, \ldots, \right\}$$

$$= 1$$

$$d_n = \frac{\Delta_n}{\Delta_{n-1}} = f \quad , \quad d_{n-1} = \cdots = d_1 = 1$$

$a_1)$ $(xI - C_f)$ in normal formu

$$D_f = \begin{bmatrix} 1 & & & & 0 \\ & 1 & & & \\ & & 1 & & \\ & 0 & & \ddots & \\ & & & & f \end{bmatrix}$$ dir.

$a_2)$ $(xI - C_f)$ in değişmez çarpanları

$d_1 = \ldots = d_{n-1} = 1$, $d_n = f$ tir.

$a_3)$ $f(x) = \underline{\underline{p_1^{e_1}}} \cdot \underline{\underline{p_2^{e_2}}} \cdots \underline{\underline{p_r^{e_r}}}$;

p_i'ler farklı asal polinomlar ve e_i ler pozitif olsun. $(xI - C_f)$ in temel bölenleri

$p_1^{e_1}, p_2^{e_2}, \ldots, p_r^{e_r}$ dir.

$D_{p_i^{e_i}} = \text{köşeg}\,[1, 1, \ldots, 1, p_i^{e_i}]$ alırsak

$a_4)$ $xI - C_f$

$$D' = \begin{bmatrix} D_{p_1^{e_1}} & & & 0 \\ & D_{p_2^{e_2}} & & \\ & & \ddots & \\ 0 & & & D_{p_r^{e_r}} \end{bmatrix} \qquad D_f$$

matrisine denk olur.

$$\Delta'_M = |D_{p_1^{e_1}}| \cdot |D_{p_2^{e_2}}| \cdots |D_{p_r^{e_r}}| = p_1^{e_1} \cdots p_r^{e_r}$$

$$\Delta'_M = f$$

$$\Delta'_{n-1} = \left\{ \begin{bmatrix} \begin{matrix} 1 & & \\ & \ddots & \\ & & p_1^{e_1} \end{matrix} & \\ & \begin{bmatrix} D_{p_2}^{e_2} & & \\ & \ddots & \\ & & D_{p_r}^{e_r} \end{bmatrix} \end{bmatrix} \right. ,$$

$$p_2^{e_2} \cdot p_3^{e_3} \cdots p_r^{e_r} = \frac{f}{p_1^{e_1}}, \quad \frac{f}{p_2^{e_2}}$$

$$\cancel{p_1^{e_1}} \cdot p_2^{e_2} \cdots p_r^{e_r}, \quad p_1^{e_1} \cancel{p_2^{e_2}} \cdots p_r^{e_r}$$

$$\frac{f}{p_3^{e_3}} \cdots$$

$$\Delta'_{n-1} = 1$$

D' matrisinin değişmez çarpanları

$$d'_m = \frac{\Delta'_m}{\Delta'_{m-1}} = f \quad , \quad d'_1 = \cdots = d'_{m-1} = 1$$

$$\frac{\Delta'_{m-1}}{\Delta'_{m-2}} \quad \begin{matrix} 1 \\ 1 \end{matrix}$$

D' değişmez çarpanları ile $xI - C_f$ in değişmez çarpanları eşit; ve bu iki matris birbirine denktir. (a_4 ispatlandı.)

b) $f_1, f_2, \ldots, f_s$; $\deg f_i = m_i \geq 1$

ve $f_{i-1} \mid f_i$:

$$R = \begin{bmatrix} C_{f_1} & & & \\ & C_{f_2} & & \\ & & \ddots & \\ & & & C_{f_s} \end{bmatrix}$$

$\deg f_1 + \deg f_2 + \ldots + \deg f_s = n$

$m_1 + m_2 + \ldots + m_s = \underline{\underline{n}}$

Yukarıdaki R matrisine Rasyonel Formdadır (Biçimdedir) denir.

(a_1) den her $(xI_{m_i} - C_{f_i})$ nin

$\text{köşeg}[1, \ldots, 1, f_i]$ matrisine denk;

bu yüzden $(x\underline{I} - C)$ nin

$\text{köşeg}[1, 1, \ldots, 1, f_1, f_2, \ldots, f_s]$

matrisine denkliğini biliyoruz.

Böylece R rasyonel formda ise,

$(x\underline{I} - R)$ nin değişmez çarpanları

$d_1 = \ldots = d_s = 1$; $d_{s+1} = f_1$, $d_{s+2} = f_2, \ldots, d_{s+(n-s)}$

$\qquad\qquad\qquad\qquad\qquad\qquad \parallel$

$\qquad\qquad\qquad\qquad\qquad\qquad d_n = f_s$

olur. $C \sim \begin{bmatrix} d_1 & & \\ & \ddots & \\ & & d_s \end{bmatrix} \sim$

$xI - C \quad xI - R \qquad R = \begin{bmatrix} C_{d_1} & & \\ & \ddots & \\ & & C_{d_s} \end{bmatrix}$

$x-a \quad \ldots \quad x-a_i$

$f_i = \underline{(p_{i1})^{k_{i1}} p_{i2}^{k_{i2}} \cdots p_{it_i}^{k_{it_i}}}$, f_i nin

asal polinomlara ayrılışı olsun.

$$\boxed{\begin{matrix} b^2-4ac<0 \\ ax^2+bx+c \end{matrix}}$$

$(x-a)^{b_i}$

Bir yandan (a_4) ile her bir

$xI_{m_i} - C_{f_i}$,

$$D_i = \begin{bmatrix} D_{p_{i1}}^{k_{i1}} & & \\ & \ddots & \\ & & D_{p_{it_i}}^{k_{it_i}} \end{bmatrix}$$

matrisine denk; ve $xI - C$,

$$D' = \begin{bmatrix} D_1 & & & \\ & D_2 & & \\ & & \ddots & \\ & & & D_s \end{bmatrix}$$ matrisine denktir.

Diğer yandan $(xI - C)$ nin

temel bölenleri $\{ \underbrace{p_{i1}^{k_{i1}}, \ldots, p_{ij}^{k_{ij}}}_{f_i} \mid \begin{array}{l} i=1,\ldots,s \\ j=1,\ldots,n_i \end{array} \}$

terimlerinin yer değiştirmesiyle elde edilir.

Böylece $(xI - C)$,

$S = \{ p_i^{e_{ij}} \ 1 \leq i \leq s, \ 1 \leq j \leq n_i \}$ temel bölenleri

olmak üzere

$$D = \begin{bmatrix} D_{p_1}^{e_{11}} & & & & & \\ & \ddots & & & & \\ & & D_{p_1}^{e_{1n_1}} & & & \\ & & & \ddots & & \\ & & & & D_{p_r}^{e_{11}} & \\ & & & & & \ddots \\ & & & & & & D_{p_r}^{e_{1n_r}} \end{bmatrix}$$

matrisine denktir.

$D_{p_i}^{e_{ij}}$, $(xI - C_{p_i^{e_{ij}}})$ nin normal formu olduğundan (a_1) ile $xI - C$ matrisi

$$R_A = \begin{bmatrix} C_{p_1}^{e_{11}} & & & & & \\ & \ddots & & & & \\ & & C_{p_1}^{e_{1n_1}} & & & \\ & & & \ddots & & \\ & & & & C_{p_r}^{e_{r1}} & \\ & & & & & \ddots \\ & & & & & & C_{p_r}^{e_{rn_r}} \end{bmatrix}$$

matrisine denktir.

$b_2)$ C matrisi ile R_A benzerdir.

Yukarıdaki yazılıştaki R_A matrisine

Asal Rasyonel Form denir

$$R = \begin{bmatrix} C_{f_1} & & & \\ & C_{f_2} & & \\ & & \ddots & \\ & & & C_{f_s} \end{bmatrix}$$

$d_1 = f_1 = p_{11}^{e_{11}} p_{12}^{e_{12}}$

$d_2 = f_2 = p_{21}^{e_{21}} \cdot p_{22}^{e_{22}}$

$\vdots$

$d_s = f_s =$

$$R_A = \begin{bmatrix} \begin{bmatrix} C_{p_1^{e_{11}}} & & \\ & C_{p_1^{e_{12}}} & \\ & & \ddots \end{bmatrix} & \\ & \ddots \end{bmatrix}$$

$e_{11} \geq e_{12} \geq \dots \geq e_{1r_1}$

Şartıyla tek türlü belirlidir

Örnek: A, $\mathbb{R}$ üzerinde kare bir matris ve

$\Delta_A = |xI - A|$ karakteristik polinomu

$\Delta_A = (x+1)^2 . (x-2)^3$ olsun. $(xI - A)$ nın

temel bölenleri aşağıdakilerden biridir:

a) $(x+1)^2, (x-2)^3$

$$\begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & (x+1)^2(x-2)^3 \end{bmatrix}$$

b) $(x+1), (x+1), (x-2)^3$

$$\begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & x+1 & \\ & & & & (x-2)^3(x+1) \end{bmatrix}$$

c) $(x+1)^2, (x-2)^2, (x-2)$

$$\begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & x-2 & \\ & & & & (x-2)^2(x+1)^2 \end{bmatrix}$$

$J_c^{(m)}$

Temel Bölenler

a) $(x+1)^2, (x-2)^3$

$$J_{(a)} = \left[\begin{array}{cc|ccc} -1 & 0 & & & \\ 1 & -1 & & 0 & \\ \hline & & 2 & 0 & 0 \\ & 0 & 1 & 2 & 0 \\ & & 0 & 1 & 2 \end{array}\right]$$

b) $(x+1)^1, (x+1)^1, (x-2)^3 \rightarrow$

$$J_{(b)} = \left[\begin{array}{c|c|ccc} -1 & & & & \\ \hline & -1 & & & \\ \hline & & 2 & 0 & 0 \\ & & 1 & 2 & 0 \\ & & 0 & 1 & 2 \end{array}\right]$$

c) $(x+1)^2, (x-2)^2, (x-2)$

$$J_{(c)} = \left[\begin{array}{cc|cc|c} -1 & 0 & & & \\ 1 & -1 & & & \\ \hline & & 2 & 0 & \\ & & 1 & 2 & \\ \hline & & & & 2 \end{array}\right]$$

temel

a) $\underbrace{(x+1)^2}, \underbrace{(x-2)^3}$

$$\begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & (x+1)^2(x-2)^3 \end{bmatrix}$$

$$\begin{bmatrix} C_{(x+1)^2} & 0 \\ 0 & C_{(x-2)^3} \end{bmatrix} = \begin{bmatrix} 0 & -1 & & & \\ 1 & -2 & & & \\ & & 0 & 0 & 8 \\ & & 1 & 0 & -6 \\ & & 0 & 1 & 12 \end{bmatrix}$$

$x^3 - 3 \cdot 4 \cdot x^2 + 3 \cdot 2 \cdot x - 8$

d) $(x+1), (x+1), (x-2), (x-2), (x-2)$

$$\begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & (x-2) & & \\ & & & (x+1)(x-2) & \\ & & & & (x+1)(x-2) \end{bmatrix}$$

e) $(x+1)^2, (x-2), (x-2), (x-2)$

$$\begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & (x-2) & & \\ & & & (x-2) & \\ & & & & (x-2)(x+1)^2 \end{bmatrix}$$

$$
R(C) = \begin{bmatrix} C_{x-2} & & 0 \\ & C_{x-2} & \\ 0 & & C_{(x-2)(x+1)^2} \end{bmatrix}
$$

$$
C = \begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & (x-2) & & \\ & & & (x-2) & \\ & & & & (x-2)(x+1)^2 \end{bmatrix}
= \left[\begin{array}{c|c|ccc} 2 & & & & \\ \hline & 2 & & & \\ \hline & & 0 & 0 & 2 \\ & & 1 & 0 & 3 \\ & & 0 & 1 & 0 \end{array}\right]
$$

$$
(x-2)(x^2+2x+1) = x^3 - 3x - 2
$$

$$\underbrace{(x-2)^1}_{p_1^{a_1}},\ \underbrace{(x-2)^1}_{p_1^{a_1}},\ \underbrace{(x-2)^1}_{p_1^{a_1}}\ \underbrace{(x+1)^2}_{p_2^{a_2}}$$

$$\left[\begin{array}{ccc|cc}
2 & 0 & 0 & & \\
0 & 2 & 0 & & \\
0 & 0 & 2 & & \\
\hline
 & & & 0 & -1 \\
 & & & 1 & -2
\end{array}\right]$$

Tanım: A, $\mathbb{R}$ üzerinde $n\times n$ bir matris olsun. $f(A)=0$ sağlayan en küçük pozitif dereceli monik f polinomuna A'nın minimal polinomu denir.

Önerme: $f(x)=a_0+a_1x+\dots+a_{n-1}x^{n-1}+x^n$ verildiğinde

$$C_f=\begin{bmatrix} 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & & 0 & -a_1 \\ 0 & 1 & \ddots & \vdots & \vdots \\ & 0 & \ddots & 0 & \vdots \\ 0 & & 0 & 1 & -a_{n-1} \end{bmatrix}=A$$

eşlikçi matrisinin minimal polinomu f'tir.

İspat: $C_f:\mathbb{R}^n\to\mathbb{R}^n$, $x\mapsto A.x$ alalım.

$$C_f(e_1) = A.e_1 = e_2$$

$$C_f(\underline{\underline{e_2}}) = A.e_2 = e_3 = \underline{A^{\underline{\underline{2}}} e_1}$$

$\vdots$

$$C_f(e_{n-1}) = A.e_{n-1} = e_n = A^{\underline{\underline{n-1}}} . e_1$$

$$C_f = \begin{bmatrix} 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & & 0 & -a_1 \\ 0 & 1 & \ddots & \vdots & \vdots \\ & 0 & \ddots & 0 & \vdots \\ 0 & & 0 & 1 & -a_{n-1} \end{bmatrix} = A \qquad \begin{matrix} 1 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{matrix} \qquad \begin{matrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{matrix}$$

$$\uparrow \ \uparrow \qquad \uparrow$$

$$C_f(e_n) = -a_0 . e_1 - a_1 . e_2 - a_2 . e_3 - \cdots - a_{n-1} e_{n-1} = A.e_n$$

$$= A^? . e_1$$

$$A^n \cdot e_1 + a_{n-1} \cdot A^{n-1} \cdot e_1 + a_{n-2} A^{n-2} \cdot e_1 + \dots + a_1 \cdot A e_1$$

$$+ a_0 e_1 = 0$$

$$\left(A^n + a_{n-1} \cdot A^{n-1} + a_{n-2} \cdot A^{n-2} + \dots + a_1 \cdot A + a_0\right) \cdot e_1 = 0$$

$$f(A) \cdot e_1 = 0$$

$$C_f(e_n) = -a_0 e_1 - a_1 \cdot e_2 - a_2 \cdot e_3 - \dots - a_{n-1} e_{n-1} = A \cdot e_n$$
$$= A^n \cdot e_1$$

$f(A) \cdot e_j = 0$ (her j için olmalı)

$$f(A) \cdot e_j = \underbrace{f(A)}\,\underbrace{A^j} \cdot e_1 = A^j \underbrace{f(A) \cdot e_1}_{0} = 0$$

Her j için $f(A) \cdot e_j = 0$

olduğundan $\underset{\substack{\Downarrow \\ C_f}}{f(A)} \equiv 0$'dır

$g(x) \in \mathbb{R}[x]$ ve $\underset{\substack{\parallel \\ d}}{der(g)} < der(f)$ ve

$g(A) = 0$ olsun.

$$g(x) = b_0 + b_1 x + \dots + b_{d-1} x^{d-1} + x^d$$

alırsak;

$$\underset{\substack{\parallel \\ 0}}{g(A)(e_1)} = b_0 e_1 + b_1 . \underbrace{A . e_1} + b_2 . A^2 e_1 + \dots + A^d e_1$$

$$= b_0 . e_1 + b_1 . e_2 + b_2 . e_3 + \dots + b_{d-1} \underbrace{(A^{d-1} . e_1)}_{e_{d-1}}$$

$$+ \underbrace{A^d . e_1}_{e_d}$$

$=$

$= 0$ olmalı.

Bu durum $\{e_1, e_2, \dots, e_d\}$ nin lineer bağımsızlığı ile çelişir.

Bu yüzden f, A nın minimal polinomudur.

Önerme: $A = \left[\begin{array}{c|c} C_{f_1} & 0 \\ \hline 0 & C_{f_2} \end{array}\right]$

f_1 ve f_2 polinomlarının eşlikçi matrislerinin blok köşegensel matrisi olsun. A nın minimal polinomu $\text{ekok}(f_1, f_2)$ dir.

$f(A) = 0$ olsun.

$f(A) = \left[\begin{array}{c|c} f(C_{f_1}) & 0 \\ \hline 0 & f(C_{f_2}) \end{array}\right]$ olur.

$$\left[\begin{array}{c|c} A & 0 \\ \hline 0 & B \end{array}\right] \cdot \left[\begin{array}{c|c} A & 0 \\ \hline 0 & B \end{array}\right] = \begin{bmatrix} A^2+0 & A.0+B0 \\ 0.A+0B & B^2+0 \end{bmatrix}$$

$$\left[\begin{array}{c|c} f(A) & \\ \hline & f(B) \end{array}\right] = 0$$

$f(C_{f_1}) = 0$ ve $f(C_{f_2}) = 0$ olur.

f in minimal olduğundan

$f_1 | f$ ve $f_2 | f$ olur.

Bu şartı sağlayan minimum dereceli f_1

ekok(f_1, f_2) olur.

Sonuç: A, $\mathbb{R}$ üzerinde $n\times n$ bir matris olsun. $(xI-A)$ karakteristik matrisinin en son değişmez çarpanı $\underline{d_n}$, A nın minimal polinomudur.

İspat: A nın Smith normal formunun değişmez çarpanları $d_1=d_2=\dots=d_s=1$, $d_{s+1}, d_{s+2}, \dots, \underline{d_n}$ olsun.

A matrisi $C = \begin{bmatrix} C_{d_s} & & & \\ & C_{d_{s+1}} & & \\ & & \ddots & \\ & & & C_{d_n} \end{bmatrix}$

rasyonel formdaki C matrisine benzerdir.

A nın (C'nin) minimal polinomu

$\text{ekok}(d_s, d_{s+1}, \ldots, d_n)$ dur.

Değişmez çarpanlar $d_s \mid d_{s+1}$, $d_{s+1} \mid d_{s+2}, \ldots,$

$d_{n-1} \mid d_n$.

sağladığından $\text{ekok}(d_s, \ldots, d_n) = d_n$ olur.

A. nın minimal polinomu d_n olur.

Örnek: A, $\mathbb{Q}$ üzerinde bir matris,

$\Delta_A = \det(xI - A) = \underline{(x+1)^2(x^3-1)}$ ve

$\min(A) = (x+1).(x^3-1)$ olduğuna göre

A nın rasyonel formunu bulunuz.

$(xI - A)$'nın değişmez çarpanları bulunmalı.

↳ $d_1, d_2, \dots, d_n$

$$\Delta_A = d_1 . d_2 \cdots d_n = \underbrace{\frac{\Delta_n}{\cancel{\Delta_{n-1}}}}_{d_n} . \underbrace{\frac{\cancel{\Delta_{n-1}}}{\Delta_{n-2}}}_{d_{n-1}} \cdots \frac{\Delta_2}{\cancel{\Delta_1}} . \underbrace{\frac{\cancel{\Delta_1}}{\Delta_0}}_{d_1}$$

$\downarrow$

$$, \quad \min(A)$$

$$1, 1, (x+1), (x+1)(x^3-1)$$

The last entry carries a handwritten "=" (written as ‖) to $\min(A)$ above it.

$d_{s-1}, d_s, d_{s+1}, \dots, d_n \Rightarrow$

$$R(A) = \begin{bmatrix} C_{d_s} & & \\ & \ddots & \\ & & C_{d_n} \end{bmatrix}$$

$d_4 = (x+1)$, $C_{d_4} = [-1]_{1\times 1}$

$d_5 = (x+1)(x^3-1)$, C_{d_5}

$d_5 = x^4 - x + x^3 - 1$

$$R(A) = \left[\begin{array}{c|cccc} -1 & & & 0 & \\ \hline & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ & 0 & 1 & 0 & 0 \\ & 0 & 0 & 1 & -1 \end{array}\right]$$

Asal Rasyonel Form İçin:

Temel Bölenler $= \{ (x+1), (x-1), (x^2+x+1), (x+1) \}$

$$\underset{\text{asal}}{R_A}\big(\overset{\text{matris}}{A}\big) = \left[\begin{array}{c|c|cc|c} -1 & 0 & 0 & 0 & 0 \\ \hline 0 & -1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ \hline 0 & 0 & 0 & 0 & -1 \end{array}\right]$$

Örnek: $C = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 1 & 1 \end{bmatrix}$ matrisinin rasy. ve asal rasy. formları?

$$xI - C = \begin{bmatrix} x-1 & -1 & 0 \\ 0 & x & -2 \\ 0 & -1 & x-1 \end{bmatrix}$$

$$\underset{C_1 \leftrightarrow C_2}{} \begin{bmatrix} -1 & x-1 & 0 \\ x & 0 & -2 \\ -1 & 0 & x-1 \end{bmatrix} \underset{\sim}{-C_1} \begin{bmatrix} 1 & x-1 & 0 \\ -x & 0 & -2 \\ 1 & 0 & x-1 \end{bmatrix}$$

$$\underset{\sim}{\substack{x.S_1+S_2 \\ -S_1+S_3}} \begin{bmatrix} 1 & x-1 & 0 \\ 0 & x(x-1) & -2 \\ 0 & -(x-1) & x-1 \end{bmatrix} \underset{\sim}{C_2 \leftrightarrow C_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & x(x-1) \\ 0 & -(x-1) & -(x-1) \end{bmatrix}$$

$$\overset{\frac{1}{2}S_2}{\sim}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{x\cdot(x-1)}{2} \\ 0 & -(x-1) & -(x-1) \end{bmatrix} \overset{(x-1)S_2+S_3}{\sim} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{x(x-1)}{2} \\ 0 & 0 & \end{bmatrix}$$

$$(x-1)\left[\frac{x}{2}(x-1)-1\right]$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & x(x-1) \\ 0 & -(x-1) & -(x-1) \end{bmatrix}$$

$$(x-1)\left(\frac{x^2-x-2}{2}\right)$$

$$\left(\frac{x^3-2x^2-x+2}{2}\right)$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & x^3-2x^2-x+2 \end{bmatrix} \longleftarrow$$

$$d_3 = x^3-2x^2-x+2$$

$$R(C) = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix} \sim C$$

Benzer

$$x^2(x-2)-(x-2)$$
$$=(x-2)(x^2-1)$$

A Benzerlik.

$xI - A$ Denklik

$$d_3 = \underbrace{x^3 - 2x^2 - x + 2}$$

$$\begin{bmatrix} C_{d_1} & & \\ & \ddots & \\ & & C_{d_n} \end{bmatrix} \leftarrow \begin{bmatrix} d_1 & & \\ & \ddots & \\ & & d_n \end{bmatrix}$$

Temel Bölenler $= x-1,\ x+1,\ x-2$

$$R_A(C) = \left[\begin{array}{c|c|c} 1 & 0 & 0 \\ \hline 0 & -1 & 0 \\ \hline 0 & 0 & 2 \end{array}\right]$$

Jordan Formu:

A) $c \in \mathbb{R}$ için $m \times m$ tipindeki

$$J_c^{(m)} = \begin{bmatrix} c & 0 & 0 & & 0 \\ 1 & c & 0 & & \vdots \\ 0 & 1 & c & & \vdots \\ & & & \ddots & 0 \\ 0 & 0 & \cdots & 1 & c \end{bmatrix}$$

matrisine c ye ait temel Jordan matrisi denir.

$$\left(x I_m - J_c^{(m)} \right) = \begin{bmatrix} x-c & 0 & 0 & & 0 \\ -1 & x-c & 0 & & \\ 0 & -1 & x-c & & \\ & & & \ddots & \\ 0 & \cdots & 0 & -1 & x-c \end{bmatrix}$$

$$\Delta_m = \left| xI_m - J_c^{(m)} \right| = (x-c)^m$$

$$\Delta_{m-1} = \text{ebob}\left\{ \underbrace{\begin{vmatrix} -1 & x-c & & \\ & -1 & \ddots & \\ & & \ddots & x-c \\ 0 & & & -1 \end{vmatrix}}_{(-1)^{m-1}} , \right.$$

$$\Delta_{m-1} = 1, \Rightarrow \Delta_{m-2} = \cdots = \Delta_1 = 1$$

$(xI_m - J_c^{(m)})$ matrisinin değişmez çarpanları

$$d_1 = d_2 = \cdots = d_{m-1} = 1, \quad d_m = (x-c)^m$$

Bu durumda $\left(xI_m - J_c^{(m)}\right)$

$$D_c^{(m)} = \begin{bmatrix} 1 & & & & \\ & 1 & & \huge 0 & \\ & & \ddots & & \\ & \huge 0 & & 1 & \\ & & & & (x-c)^m \end{bmatrix}$$

matrisine denk olur.

C nin temel bölenleri kümesi

$$\{(x-c_i)^{e_{ij}} : 1 \leq i \leq r, \ 1 \leq j \leq n_i\}$$

ise C nin asal rasyonel formu

$$\underbrace{(x-c_i)^{e_{ij}}}\, = c_{ij}$$

$$\begin{bmatrix} C_{c_{11}} & & & & & & \\ & C_{c_{12}} & & & & & \\ & & \ddots & & & & \\ & & & C_{c_{1n_1}} & & & \\ & & & & \ddots & & \\ & & & & & C_{c_{r1}} & \\ & & & & & & \ddots \\ & & & & & & & C_{c_{rn_r}} \end{bmatrix}$$

$\left(xI - C_{c_{ij}}\right)$ değişmez çarpanları $1, 1, \ldots, 1,$

$(x-c_i)^{e_{ij}}$

$\left(xI - J_c^{e_{ij}}\right)$ nin de değişmez çarpanları

$1, 1, \ldots, 1, (x-c_i)^{e_{ij}}$

$\left(xI - C_{c_{ij}}\right) \sim \left(xI - J_c^{(e_{ij})}\right)$ denktir

$C \sim C_{ij} \sim J_c^{e_{ij}}$ benzer olur.

Teorem: $n\times n$ tipindeki A matrisi için $\Delta_A = |xI - A|$, $\mathbb{R}$ üzerinde lineer polinomların çarpımı ise, A Jordan formundaki bir matrise benzerdir.

$$R = \begin{bmatrix} \boxed{\begin{matrix} J_{c_1}^{(e_{11})} & & \\ & \ddots & \\ & & J_{c_1}^{(e_{1n_1})} \end{matrix}} & & \\ & \ddots & \\ & & \boxed{\begin{matrix} J_{c_r}^{e_{r1}} & & \\ & \ddots & \\ & & J_{c_r}^{e_{rn_r}} \end{matrix}} \end{bmatrix}$$