$$\cancel{\begin{bmatrix} 1 \\ x - x^3 \end{bmatrix}}$$

Doğrusu

$$\begin{bmatrix} 1 \\ 1 \cdot x^3 - x \end{bmatrix}$$

$\downarrow$

monik

Sonuç:

b) $\mathbb{R}[x]$ üzerinde $n \times n$ tipinde A ve B nin denk olması için G.Y. Şart $B = P.A.Q$ olacak şekilde $\mathbb{R}[x]^{n \times n}$ P ve Q tersinir matrisler olmasıdır.

a) $\mathbb{R}[x]$ üzerinde $n \times n$ A matrisinin tersinir olması için G.Y. Şart A nın elementer matrislerin bir çarpımı olmasıdır.

c) Denk matrislerin normal formu eşittir.

d) $\mathbb{R}[x]$ üzerinde $n \times n$ tipindeki her A için, A ve A^T nun normal formları eşittir.

İspat: a $\Rightarrow$ (A tersinir olsun).

A nın normal formu D_A olsun.

$D_A = \text{köşeg}[d_1, d_2, \ldots, d_r, 0, \ldots, 0]$

Her i için $d_i \mid d_{i+1}$ ve d_i ler monik.

A ve D_A denk olduklarından öyle elementer $E_1, \dots, E_k, F_1, \dots, F_s$ matrisleri vardır ki

$$D_A = E_k \dots E_1 . A . F_1 \dots F_s$$

$1 . x^a$

$c . 1$

monik

1

Her iki tarafın determinantıyla

$$|D_A| = \underbrace{|E_k| \dots |E_1|}_{\neq 0} . \underbrace{|A|}_{\neq 0} . \underbrace{|F_1| \dots |F_s|}_{\neq 0}$$

$\Rightarrow |D_A| = d_1 . d_2 \dots d_r$ ve $|D_A| \neq 0$.

$d_1 | d_2 | \dots | d_r$ d_i monik. $\Rightarrow d_i = 1$

$$A = E_1^{-1} \ldots E_k^{-1} \overset{\downarrow \text{DA}}{I} \cdot F_s^{-1} \ldots F_1^{-1} \rightarrow$$ Elem. matrisl. çarpımı.

E_j^{-1} ve F_k^{-1} birer elementer matris.

($\Leftarrow$) A elem. matrislerin çarpımı ise;

$A = E_1 . E_2 \ldots E_k$ ise

$$(E_1 E_2 \ldots E_k)^{-1} = E_k^{-1} \ldots E_1^{-1} = A^{-1}$$

dolayısıyla A da tersinirdir

b) $\Rightarrow$ A ve B denk olsun.

$$B = \underbrace{E_1 \cdots E_k}_{P} A . \underbrace{F_1 \cdots F_s}_{Q}$$

olacak şekilde E_j, F_ℓ elementer matrisler vardır.

$$B = P . A . Q$$

P ve Q elem. matrisl. çarpımı old.dan tersinir-dir

($\Leftarrow$) P ve Q tersir olmak üzere $B = P . A . Q$ olsun

(a)'dan; P ve Q elem. matrislerin çarpımıdır.

$$B = E_1 \cdots E_k . A . F_1 \cdots F_s \Rightarrow B \overset{\text{denk}}{\sim} A$$

c) A ve B denk olsun.

$\underline{B = P \cdot A \cdot Q}$ olacak şekilde P ve Q

tersinir matrisler var.

B'nin Smith Formu D_B olsun.

B ye uygulanacak elem. işlemlerle

$\underline{R} \cdot B \cdot \underline{S} = D_B$ elde edilir. $B \sim D_B$

$R\,B\,S = \underbrace{RP} \cdot A \cdot \underbrace{QS} = D_B$

$A \sim D$

$A \sim D_B$

Smith Form un tekliğinden $D_B = D_A$

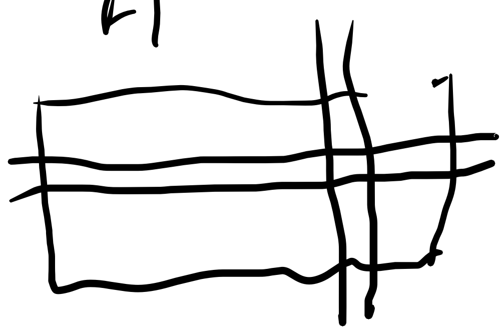
$\left(d_i = \dfrac{\Delta_i}{\Delta_{i-1}}\right)$

d) A $n \times n$ $R[x]$ üzerinde bir matris ise;

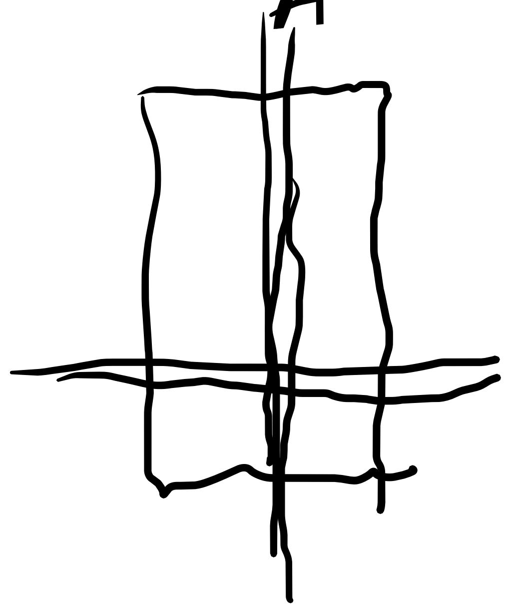
A nın $i \times i$ bir alt matrisi K_i ise;

K_i, A^T un $i \times i$ tipinde bir $L_i = K_i^T$ alt matrisine denk gelir. Bu durumda

A



A^T



$\Delta_i(A) =$ A nın $\underline{i \times i}$ minörlerinin ebobu

$=$ A^T un $\underline{i \times i}$ minörlerinin ebobu.

$$D_A = [d_1, d_2, \dots, d_r, 0, \dots, 0] = D_{A^T}$$

$$d_i = \frac{\Delta_i(A)}{\Delta_{i-1}(A)} = \frac{\Delta_i(A^T)}{\Delta_{i-1}(A^T)}$$

Örnek: $A = \left[\begin{array}{ccc} x^2 & x+1 & 0 \\ x^2-1 & x+1 & 0 \\ 0 & 0 & 3(x+1)^2 \end{array}\right]$ S.N. Formu = ?

1. $x^2 + (-1)(x^2-1) = 1$

$-S_2 + S_1 \to S_1$ $\left[\begin{array}{ccc} 1 & 0 & 0 \\ \boxed{x^2-1} & x+1 & 0 \\ 0 & 0 & 3(x+1)^2 \end{array}\right]$

$(1-x^2)S_1 + S_2 \to S_2$ $\sim \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & x+1 & 0 \\ 0 & 0 & 3.(x+1)^2 \end{array}\right] \underset{\frac{1}{3}.S_3}{\sim} \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & x+1 & 0 \\ 0 & 0 & (x+1)^2 \end{array}\right]$

Değişmez çarpanlar: 1, $x+1$, $(x+1)^2$

Temel bölenler: $(x+1)$, $(x+1)^2$

Örnek:

$$A=\begin{bmatrix} 2x+1 & x & -x & x+1 \\ x-1 & x-1 & 0 & x-1 \\ 0 & 0 & x^2-1 & 0 \\ x-1 & x-1 & x^2-1 & x-1 \end{bmatrix}, \quad D_A=?$$

$$2.C_3+C_1 \to C_1 \qquad \sim \begin{bmatrix} 1 & x & -x & x+1 \\ x-1 & x-1 & 0 & x-1 \\ 2x^2-2 & 0 & x^2-1 & 0 \\ 2x^2+x-3 & x-1 & x^2-1 & x-1 \end{bmatrix}$$

$$\begin{array}{l} (-x)C_1+C_2 \\ x\,C_1+C_3 \\ -(x+1)C_1+C_4 \end{array} \qquad \begin{bmatrix} 1 & 0 & 0 & 0 \\ x-1 & 2x-x^2-1 & x^2-x & (x-1)(-x-1+1) \\ 2x^2-2 & -2x^3+2x & 2x^3+x^2-2x-1 & (2x^2-2)(-x-1) \\ 2x^2+x-3 & -2x^3-x^2+4x-1 & 2x^3+2x^2-3x-1 & -(2x^2+x-3)(x+1)+x-1 \end{bmatrix}$$

$$
\begin{bmatrix}
1 & 0 & 0 & 0 \\
x-1 & 2x-x^2-1 & x^2-x & (x-1)(-x-1+1) \\
2x^2-2 & -2x^3+2x & 2x^3+x^2-2x-1 & (2x^2-2)(-x-1) \\
2x^2+x-3 & -2x^3-x^2+4x-1 & 2x^3+2x^2-3x-1 & -(2x^2+x-3)(x+1)+x-1
\end{bmatrix}
$$

$$
\sim \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & -(x-1)^2 & x(x-1) & -x(x-1) \\
0 & -2x(x^2-1) & (2x+1)(x^2-1) & -2(x^2-1)(x+1) \\
0 & -(2x^3+x^2-4x+1) & 2x^3+2x^2-3x-1 & -(x+1)\cdot(2x+3)\cdot(x-1)+(x-1)
\end{bmatrix}
$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -(x-1)^2 & x(x-1) & -x(x-1) \\ 0 & -2x(x^2-1) & (2x+1)(x^2-1) & -2(x^2-1)(x+1) \\ 0 & -(2x^3+x^2-4x+1) & 2x^3+2x^2-3x-1 & -\left[(x+1)(2x+3)\cdot(x-1)\right]+(x-1) \end{bmatrix}$$

$$\underset{\sim}{C_1+C_2} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & x-1 & x(x-1) & -x.(x-1) \\ 0 & (x^2-1) & (2x+1)(x^2-1) & -2(x^2-1)(x+1) \\ 0 & x^2+x-2 & 2x^3+2x^2-3x-1 & -(x^2-1)(2x+3)+(x-1) \end{bmatrix}$$

$$\begin{array}{c} \ldots_3 + C_2 \\ \sim \end{array} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & x-1 & x(x-1) & -x.(x-1) \\ 0 & (x^2-1) & (2x+1)(x^2-1) & -2(x^2-1)(x+1) \\ 0 & x^2+x-2 & 2x^3+2x^2-3x-1 & -(x^2-1)(2x+3)+(x-1) \end{bmatrix}$$

$$\begin{array}{c} -x.C_2+C_3 \\ xC_2+C_4 \\ \sim \end{array} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & x-1 & 0 & 0 \\ 0 & (x^2-1) & (x+1).(x^2-1) & (-x-2)(x^2-1) \\ 0 & (x+2)(x-1) & x^3+x^2-x-1 & -x^3-2x^2+x+2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & x-1 & 0 & 0 \\ 0 & (x^2-1) & (x+1)\cdot(x^2-1) & (-x-2)(x^2-1) \\ 0 & (x+2)(x-1) & x^3+x^2-x-1 & -x^3-2x^2+x+2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & x-1 & 0 & 0 \\ 0 & 0 & (x+1)(x^2-1) & -(x+2)(x^2-1) \\ 0 & 0 & (x^2-1)(x+1) & (x+2)(1-x^2) \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & x-1 & 0 & 0 \\ 0 & 0 & (x+1)(x^2-1) & -(x+2)(x^2-1) \\ 0 & 0 & (x^2-1)(x+1) & (x+2)(1-x^2) \end{bmatrix}$$

$$\underset{C_4+C_3\to C_3}{\sim} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & x-1 & 0 & 0 \\ 0 & 0 & -(x^2-1) & -(x+2)(x^2-1) \\ 0 & 0 & -(x^2-1) & -(x+2)(x^2-1) \end{bmatrix}$$

$$
\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & x-1 & 0 & 0 \\ 0 & 0 & -(x^2-1) & -(x+2)(x^2-1) \\ 0 & 0 & -(x^2-1) & -(x+2)(x^2-1) \end{bmatrix}
$$

$$
\underset{\sim}{-C_3 \to C_3} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & x-1 & 0 & 0 \\ 0 & 0 & (x^2-1) & -(x+2)(x^2-1) \\ 0 & 0 & (x^2-1) & -(x+2)(x^2-1) \end{bmatrix}
$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & x-1 & 0 & 0 \\ 0 & 0 & (x^2-1) & -(x+2)(x^2-1) \\ 0 & 0 & (x^2-1) & -(x+2)(x^2-1) \end{bmatrix}$$

$$-S_3+S_4 \to S_4$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & x-1 & 0 & 0 \\ 0 & 0 & x^2-1 & -(x+2)(x^2-1) \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & x-1 & 0 & 0 \\ 0 & 0 & x^2-1 & -(x+2)(x^2-1) \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\underset{(x+2)C_2+C_3}{\sim} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & x-1 & 0 & 0 \\ 0 & 0 & x^2-1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = D_A$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & x-1 & 0 & 0 \\ 0 & 0 & x^2-1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = D_A$$

Değişmez çarpanlar: $1, \; x-1, \; x^2-1$

Temel bölenler: $x-1, \; x-1, \; x+1$