

Örnek:

$$A = \begin{bmatrix} x^2 & x+1 & 0 \\ x^2-1 & x+1 & 0 \\ 0 & 0 & 3(x+1)^2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & x+1 & 0 \\ 0 & 0 & (x+1)^2 \end{bmatrix} = \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix}$$

$$d_1 = \frac{\Delta_1}{\Delta_0}, \quad d_2 = \frac{\Delta_2}{\Delta_1}, \quad d_3 = \frac{\Delta_3}{\Delta_2}$$

$\Delta_i = A$ nin i xi minörlerinin e.b.o.b u.

$$\Delta_1 = \text{ebob} \left\{ \underline{x^2}, x+1, \underline{x^2-1}, 3(x+1)^2 \right\} = 1$$

$$\Delta_0 = 1$$

$$\Delta_2 = \text{ebob} \left\{ \underbrace{\begin{vmatrix} x^2 & x+1 \\ x^2-1 & x+1 \end{vmatrix}}_{(x+1)}, \begin{vmatrix} x^2 & 0 \\ x^2-1 & 0 \end{vmatrix}, \begin{vmatrix} x+1 & 0 \\ x+1 & 0 \end{vmatrix} \right\}$$

$$A = \begin{bmatrix} x^2 & x+1 & 0 \\ x^2-1 & x+1 & 0 \\ 0 & 0 & 3(x+1)^2 \end{bmatrix}$$

$$\Delta_2 \text{ devam. } \begin{vmatrix} x^2 & x+1 \\ 0 & 0 \end{vmatrix}, \begin{vmatrix} x^2 & 0 \\ 0 & 3(x+1)^2 \end{vmatrix}, \begin{vmatrix} x+1 & 0 \\ 0 & 3(x+1)^2 \end{vmatrix}$$

$$\underbrace{\begin{vmatrix} x+1 & 0 \\ 0 & 3(x+1)^2 \end{vmatrix}}_{(x+1)}, \begin{vmatrix} x^2-1 & x+1 \\ 0 & 0 \end{vmatrix}, \underbrace{\begin{vmatrix} x^2-1 & 0 \\ 0 & 3(x+1)^2 \end{vmatrix}}_{(x+1)}$$

$$\Delta_0 = 1, \Delta_1 = 1, \Delta_2 = x+1, \Delta_3 = 3(x+1)^2$$

$$(\cancel{x^2+x^2} - (\cancel{x^2+x^2} - x - 1))$$

$$\Delta_3 = 3 \cdot (x+1)^2 (x+1) \cdot 1$$

$$d_3 = \frac{\Delta_3}{\Delta_2} = \frac{(x+1)^3}{(x+1)} = (x+1)^2$$

$$\begin{array}{l} 4 \times 4 \\ 1 + \binom{4}{3} \binom{4}{3} \\ + \binom{4}{2} \binom{4}{2} + \binom{4}{1}^2 \end{array}$$

$$A \sim D_A = \begin{bmatrix} 1 & \text{circle} \\ \text{circle}^{x+1} & (x+1)^2 \end{bmatrix}$$

Matris Katsayılı Polinomlar:

$$M \in \mathbb{R}[x]^{3 \times 3}$$

ve

$$M = \begin{bmatrix} x+1 & -x^2+2x & 0 \\ -5+x^3 & x^3+x^2 & x-1 \\ 0 & 1-x^3 & x^2+2x \end{bmatrix}$$

$$\begin{aligned} M = & 1 \cdot \begin{bmatrix} 1 & 0 & 0 \\ -5 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} + x \cdot \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 2 \end{bmatrix} \\ & + x^2 \cdot \begin{bmatrix} 0 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + x^3 \cdot \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & -1 & 0 \end{bmatrix} \end{aligned}$$

Lemma 0: $\mathbb{R}^{n \times n}$ de bir A matrisi ve
her i için $B_i \in \mathbb{R}^{n \times n}$ olmak üzere

$$\underline{B = B_0 + B_1 x + \dots + B_d x^d}$$

matrisi verildiğinde

$$B = (xI - A) \cdot Q + R, \quad R \in \mathbb{R}^{n \times n}$$

olarak seçilirse tek türlü belirlir $Q \in \mathbb{R}[x]$

ve R vardır.

Özel olarak

$$B = (x \cdot I - A) \cdot Q \quad \text{işte}$$

$$B_0 + B_1 A + B_2 A^2 + \dots + B_d A^d = 0$$

olur

Lemma: A ve B_i \mathbb{R} üzerindeki $n \times n$ matrisler

olsun.

karakt. matris \leftarrow
$$(xI - B) = P \cdot (xI - A) \cdot Q, \quad P, Q \in \mathbb{R}[x]^{n \times n}$$

olsun. Eğer

$$P = P_0 + x \cdot P_1 + \dots + x^r \cdot P_r, \quad P_i \in \mathbb{R}^{n \times n}$$

ise;

$$(i) \tilde{P} = P_0 + B \cdot P_1 + B^2 \cdot P_2 + \dots + B^r \cdot P_r$$

tersinirdir.

$$\boxed{P^{-1} A P = B}$$

$$(ii) \underline{\tilde{P} \cdot A \cdot \tilde{P}^{-1} = B} \quad \text{dir.}$$

İspat: (i) - a) $\tilde{P} \cdot A = B \cdot \tilde{P}$ dir.

P ve $Q \in \mathbb{R}[x]^{n \times n}$ tersinir matrislerdir:

$$xI - B = P \cdot (xI - A) \cdot Q$$

$$\underbrace{|xI - B|}_{n. \text{ derece monik polinom}} = \underbrace{|P|}_{n. \text{ der. monik}} \cdot \underbrace{|xI - A|}_{n. \text{ der. monik}} \cdot \underbrace{|Q|}_{n. \text{ der. monik}}$$

n . derece monik
polinom

n . der.
monik

$$\Rightarrow |P| \cdot |Q| = 1 \Rightarrow |P| \neq 0 \vee |Q| \neq 0$$

Böylece P ve Q tersinirdir.

$$(xI - B) = P \cdot (xI - A) \cdot Q$$

$$\underline{P(xI - A)} = (xI - B) \cdot Q^{-1}$$

$$x \cdot P - P \cdot A = \underbrace{x \cdot P - \sum_{i=0}^r P_i \cdot x^i \cdot A}_{(xI - B) \cdot Q^{-1}} = \underline{(xI - B) \cdot Q^{-1}}$$

$$\stackrel{\text{Lem 0}}{\Rightarrow} B \cdot \left(\sum_{i=0}^r P_i B^i \right) - \sum_{i=0}^r P_i \cdot B^i \cdot A = 0$$

$$\boxed{B \cdot \tilde{P} = \tilde{P} \cdot A} \leftarrow$$

$$B \tilde{P} \cdot A = \tilde{P} A^2$$

$$B \cdot B \cdot \tilde{P} = \tilde{P} \cdot A^2$$

$$\boxed{B^2 \cdot \tilde{P} = \tilde{P} \cdot A^2}$$

her j sein

$$B^j \tilde{P} = \tilde{P} \cdot A^j$$

$$B \cdot \left(\sum_{i=0}^{\infty} P_i B^i \right) - \sum_{i=0}^{\infty} P_i \cdot B^i \cdot A = 0$$

P ve Q nun tersinir oldugu, görüldü;

$$\tilde{P} = P_0 + P_1 x + P_2 x^2 + \dots + P_k x^k \text{ olsun.}$$

$$I = P \cdot P^{-1} = \left(\sum_{i=0}^r x^i P_i \right) \left(\sum_{j=0}^k x^j P_j' \right)$$

$$= \sum_{i=0}^r \sum_{j=0}^k x^{i+j} P_i P_j' = I$$

$$\tilde{P} A^j = B^j \cdot \tilde{P} \quad \text{uygulanarak}$$

$$\begin{aligned} \tilde{P} \cdot \left(\sum_{j=0}^k A^j P_j' \right) &= \sum_{j=0}^k \tilde{P} \cdot A^j P_j' = \\ &= \sum_{j=0}^k B^j \cdot \tilde{P} \cdot P_j' = \sum_{j=0}^k B^j \left(\sum_{i=0}^r B^i P_i \right) \cdot P_j' \\ &= \sum_{j=0}^k \sum_{i=0}^r B^{j+i} P_i \cdot P_j' = I \end{aligned}$$

$\Rightarrow \tilde{P}$ tersinin çıkar.

(ii) için $\tilde{P} A = B \cdot \tilde{P}$ bulunmuştur.

$$\tilde{P} \cdot A \cdot \tilde{P}^{-1} = B \text{ olur.}$$

Örnek: 1) $\mathbb{Q}[x]$ te aşağıda verilen her bir matrisi normal formuna indirgeyiniz. Değirmez çarpınlarını ve temel bölünlerini bulunuz.

a) köşeg $[x^2-1, 1, x-1, 2]$

$$= \begin{bmatrix} x^2-1 & 0 & 0 & 0 \\ 0 & \textcircled{1} & 0 & 0 \\ 0 & 0 & x-1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \xrightarrow{S_1 \leftrightarrow S_2} \sim \begin{bmatrix} 0 & 1 & 0 & 0 \\ x^2-1 & 0 & 0 & 0 \\ 0 & 0 & x-1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

$$C_1 \leftrightarrow C_7 \quad \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & x^2-1 & 0 & 0 \\ 0 & 0 & x-1 & 0 \\ 0 & 0 & 0 & 2 \end{array} \right] \xrightarrow{S_2 \leftrightarrow S} \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & x-1 & 0 \\ 0 & x^2-1 & 0 & 0 \end{array} \right]$$

$$\left[\begin{array}{cccc} 0 & 1 & 0 & 0 \\ x^2-1 & 0 & 0 & 0 \\ 0 & 0 & x-1 & 0 \\ 0 & 0 & 0 & 2 \end{array} \right]$$

$$C_2 \leftrightarrow C_4 \quad \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & x-1 & 0 \\ 0 & 0 & 0 & x^2-1 \end{array} \right] \xrightarrow{\frac{1}{2} \cdot C_2} \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & x-1 & 0 \\ 0 & 0 & 0 & x^2-1 \end{array} \right]$$

$$d_1 = d_2 = 1, \quad d_3 = x-1, \quad d_4 = x^2-1$$

$$d_1 | d_2, \quad d_2 | d_3, \quad d_3 | d_4$$

$$d_1 = d_2 = 1, \quad d_3 = x-1, \quad d_4 = x^2-1$$

A' 'nin değişmez çarpanları

$\{x-1, x-1, x+1\}$ temel bölenleri.

Örnek: $A = \ker [x-1, x+1, x-1]$

$$A = \begin{bmatrix} x-1 & 0 & 0 \\ 0 & x+1 & 0 \\ 0 & 0 & x-1 \end{bmatrix} \xrightarrow{S_2 + S_1 \rightarrow S_1} \begin{bmatrix} x-1 & x+1 & 0 \\ 0 & x+1 & 0 \\ 0 & 0 & x-1 \end{bmatrix}$$

$$\begin{aligned} & \xrightarrow{-C_2 + C_1} \begin{bmatrix} -2 & x+1 & 0 \\ -x-1 & x+1 & 0 \\ 0 & 0 & x-1 \end{bmatrix} \xrightarrow{-\frac{1}{2}S_1} \begin{bmatrix} 1 & -\frac{x-1}{2} & 0 \\ -x-1 & x+1 & 0 \\ 0 & 0 & x-1 \end{bmatrix} \\ & \xrightarrow{(x+1)S_1 + S_2} \begin{bmatrix} 1 & -\frac{x-1}{2} & 0 \\ 0 & (x+1) - \frac{(x+1)^2}{2} & 0 \\ 0 & 0 & x-1 \end{bmatrix} \xrightarrow{2S_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 \cdot (x+1) - (x+1)^2 & 0 \\ 0 & 0 & x-1 \end{bmatrix} \end{aligned}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & (x+1)(2-x-1) & 0 \\ 0 & 0 & x-1 \end{bmatrix} \xrightarrow{-s_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & (x-1)(x+1) & 0 \\ 0 & 0 & x-1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \boxed{2 \cdot (x+1)} & 0 \\ 0 & 0 & x-1 \end{bmatrix}$$

$$\begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix}$$

d_i les monite $d_i \mid d_{i+1}$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & x-1 & 0 \\ 0 & 0 & x^2-1 \end{bmatrix}$$

Degirmez carpanlar: $1, (x-1), (x^2-1)$

Temel Bölenerler : $x-1, x-1, x+1$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & x-1 & 0 \\ 0 & 0 & x^2-1 \end{bmatrix}$$

Örnek: $A = \begin{bmatrix} x & 0 & -2 \\ -1 & x & -1 \\ 0 & -1 & x-2 \end{bmatrix}$

$-s_2 \rightarrow s_2$ $\begin{bmatrix} x & 0 & -2 \\ \textcircled{1} & -x & 1 \\ 0 & -1 & x-2 \end{bmatrix} \xrightarrow{s_1 \leftrightarrow s_2} \begin{bmatrix} 1 & -x & 1 \\ x & 0 & -2 \\ 0 & -1 & x-2 \end{bmatrix}$

$x \cdot C_1 + C_2$
 \sim
 $-C_1 + C_3$ $\begin{bmatrix} 1 & 0 & 0 \\ \boxed{x} & x^2 & -2-x \\ 0 & -1 & x-2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & x^2 & -2-x \\ 0 & -1 & x-2 \end{bmatrix}$

$$-S_1 \rightarrow S_3 \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & x^2 & -2-x \\ 0 & 1 & 2-x \end{bmatrix} \quad S_2 \leftrightarrow S_3 \quad \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 2-x & 0 \\ 0 & x^2 & -2-x & 0 \end{array} \right]$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & x^2 & -2-x \\ 0 & -1 & x-2 \end{bmatrix}$$

$$(x-2)C_2 + C_3 \quad \sim \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & x^2 & -2-x+x^2(x-2) \end{bmatrix}$$

$$\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & x^3 - 2x^2 - x - 2 \end{array}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & x^2 & -2 - x + x^2(x-2) \end{bmatrix}$$

$$\sim \begin{bmatrix} \textcircled{1}^{d_1} & 0 & 0 \\ 0 & \textcircled{1}^{d_2} & 0 \\ 0 & 0 & \textcircled{x^3 - 2x^2 - x - 2}^{d_3} \end{bmatrix}$$

değişmez
sarıpolar

Örnek : $A = \begin{bmatrix} 1 & 0 & x & -1 \\ x & 1 & x^2 & -x \\ 0 & 0 & x-1 & 0 \\ x^2 & 0 & x^3 & -1 \end{bmatrix}$, $D_A = ?$

$-x \cdot S_1 + S_2$
 $A \sim$
 $-x^2 \cdot S_1 + S_4$

$$\begin{bmatrix} 1 & 0 & x & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & x-1 & 0 \\ 0 & 0 & 0 & x^2-1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & x-1 & 0 \\ 0 & 0 & 0 & x^2-1 \end{bmatrix} = \underline{\underline{D_A}}$$

$$\begin{bmatrix} 1 & 0 & x & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & x-1 & 0 \\ 0 & 0 & 0 & x^2-1 \end{bmatrix}$$

$\xrightarrow{-x}$ $\xrightarrow{1}$

Örnek: 7) aşağıdakilerden hangileri $\mathbb{R}[x]$ üzerinde bir matrisin değişmez çarpanlarıdır?

a) $1, x, x, x^2, x^2+x, x^3-x^2$

$$\begin{bmatrix} d_1 & & \\ & d_2 & \\ & & \ddots \\ & & & d_r \end{bmatrix}$$

d_i : monik ve
 $d_i \mid d_{i+1}$

$$\begin{bmatrix} 1 & & & \\ & x & & \\ & & x & \\ & & & x^2 \end{bmatrix}$$

~~$x^2 \mid x^2+x$~~ olduğundan (a) da verilenler bir mat. değişmez çarpanları değildir.

$$b) \quad 1, 1, x, x, x^2, x^3 - x^2, x^4 - x^3$$

1

1

x

x

x^2

$$x^3 - x^2$$

$$x^4 - x^3$$



$$c) 1, 1, -x, x^2, x^2 - x^3$$

Terimler birbirini bölmesine rağmen
 monik olmayan terimler bulunduğundan,
 bir matrisin değmez çarpanları değildir

$$d) x^4 + 1 \quad \checkmark$$

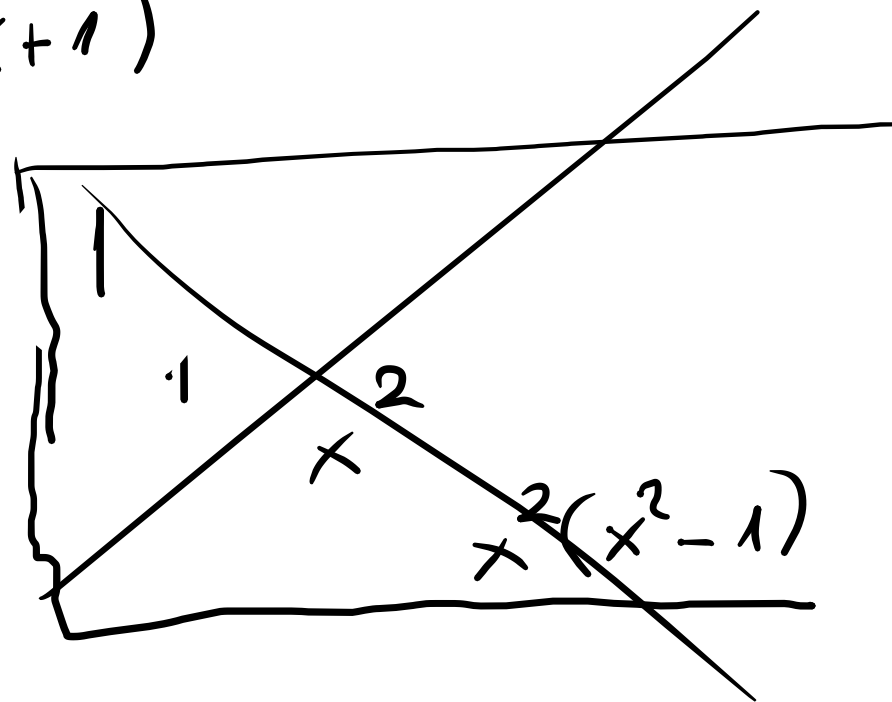
Örnek: Aşağıdakilerden hangileri sıfırdan farklı 4 değimez çarpma sahip bir matrisin temel böleneridir?

a) $x^2, x, (x-1)^2, (x-1)^2, x+1$

1 1 x

$x^2(x-1)^2(x-1)^2(x+1)$

b) $\underbrace{x^2}_{(x-1)}, \underbrace{x^2}_{(x+1)}, \underbrace{(x^2-1)}_{(x-1)(x+1)}$



$$x-5$$

$$x^2-25$$

$$\swarrow \searrow$$
$$(x-5), (x+5)$$

$$\underline{(x^2-25)^4} = (x-5)^4 \cdot (x+5)^4$$

$$(x-5)^4, (x+5)^4$$